FACULTY OF SCIENCE
M. Sc. I – Semester Examination, December 2015
Subject : Maths / Applied Maths / Maths with Computer Science
Paper – II : Real Analysis

Time : 3 Hours
Max. Marks: 80

Note : Answer all questions from Part – A and Part – B. Each question carries 4 marks in Part – A and 12 marks in Part – B.

PART – A (8 x 4 = 32 Marks)
(Short Answer Type)
1. If \( \Sigma a_n \) converges absolutely then prove that \( \Sigma a_n \) converges and also show that converse need not be true by an example.
2. If \( f \) is a continuous mapping of a compact metric space \( X \) into a metrics space \( Y \) then prove that \( f(x) \) is compact.
3. If \( p^* \) is a refinement of partition \( p \) then prove that
   \[ U(p^*, f, \alpha) \leq U(p, f, \alpha). \]
4. If \( f \in R(\alpha) \) and \( g \in R(\alpha) \) \( [a, b] \) then prove that \( fg \in R(\alpha) \).
5. If \( \lim_{n \to \infty} f_n(x) = f(x) \) and if \( M_n = \sup_{x \in E} |f_n(x) - f(x)| \) then prove that \( f_n \) converge to \( f \) uniformly on \( E \) if and only if \( M_n \) converge to zero.
6. If \( f_n(x) = n^2(1-x^2)^n \), \( 0 \leq x \leq 1 \), \( n = 1, 2, \ldots \), then show that
   \[ \lim_{n \to \infty} \frac{1}{n} \int_0^1 f_n(x) dx = \int_0^1 \lim_{n \to \infty} \frac{1}{n} f_n(x) dx. \]
7. If a vector space \( X \) is spanned by a set of \( r \) vectors then show that \( \dim X \leq r \).
8. Prove that a linear operator \( A \) on a finite dimensional vector space \( X \) is one to one if and only if the range of \( A \) is all of \( X \).

PART – B (4 x 12 = 48 Marks)
(Essay Answer Type)
9. (a) If \( f \) is a continuous mapping of a compact metric space \( X \) into metric space \( Y \) then prove that \( f \) is uniformly continuous on \( X \).

   OR

(b) If \( f \) is a continuous mapping of metric space \( X \) into a metric space \( Y \) and if \( E \) is connected subset of \( X \) then prove that \( f(E) \) also connected also prove that if \( f(a) < f(b) \) on the interval \( [a, b] \) and if \( f(a) < c < f(b) \) then \( \exists \) a point \( x \in (a, b) \) \( \exists \) \( f(x) = c \).

10. (a) If \( \alpha \) increases monotonically and \( \alpha' \in R \) on \( [a, b] \) and if \( f \) is bounded on \( [a, b] \) then prove that \( f \in R(\alpha) \) if an only if \( f \alpha' \in R \) and also show that
    \[ \int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx. \]

   OR

(b) If \( f_1, f_2 \in R(\alpha) \) on \( [a, b] \) and \( c \) is any constant then prove that \( f_1 + f_2 \in R(\alpha) \) and \( cf_1 \in R(\alpha) \) and also show that
    \[ \int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \text{ and } \int_a^b c f d\alpha = c \int_a^b f d\alpha. \]
11 (a) If \( \{f_n\} \) is a sequence of function differentiable on \([a, b]\) and if \([f_n(x_0)]\) converge for some point \(x_0\) and \([a, b]\) and also if \([f'_n]\) converge uniformly on \([a, b]\) then prove that \([f_n]\) converge uniformly on \([a, b]\) to a function \(f\) and also show that \(f'(x) = \lim_{n \to \infty} f'_n(x)\), \(a \leq x \leq b\).

OR

(b) State and prove Weierstrass approximation theorem.

12 (a) If \(E\) is an open set in \(\mathbb{R}^n\), \(f\) maps \(E\) into \(\mathbb{R}^m\), \(f\) is differentiable at \(x_0 \in E\), \(g\) maps an open set contains \(f(E)\) into \(\mathbb{R}^k\) and \(g\) is differentiable at \(f(x_0)\) then prove that the mapping \(F\) of \(E\) into \(\mathbb{R}^k\) defined by \(F(x) = g(f(x))\) is differentiable at \(x_0\) and \(R(F(x_0)) = g'(f(x_0)) f'(x_0)\).

OR

(b) Define contraction mapping and also state and prove contraction principle.