FACULTY OF SCIENCE
M.Sc. II-Semester Examination, May / June 2016
Subject: Mathematics / Applied Mathematics
Paper - II
Advanced Real Analysis

Time : 3 hours

Max. Marks : 80

Note: Answer all questions from Part-A and Part-B. Each question carries 4 marks in Part-A and 12 marks in Part-B.

PART – A (8 x 4 = 32 Marks)
(Short Answer Type)

1. Prove that if \( m^*(A) = 0 \) then \( m^*(A \cup B) = m^*(B) \).
2. Prove that if \( f \) is a measurable function and \( f = g \) a.e. then \( g \) is measurable.
3. Show that if
   \[
   f(x) = \begin{cases} 
   0 & \text{if } x \text{ is irrational} \\
   1 & \text{if } x \text{ is rational}
   \end{cases}
   \]
   then \( \int_b^a f(x) \, dx = b - a \) and \( \int_b^a f(x) \, dx = 0 \).
4. Show that if \( f \) is integrable over \( E \), then so is \( |f| \) and \( \left| \int_E f \right| \leq \int_E |f| \).
5. Show that \( D^*(f(x)) = -D^*_c f(x) \).
6. Prove that every convergent sequence is a Cauchy sequence.
7. Consider the mapping \( f = (f_1, f_2) \) of \( \mathbb{R}^3 \) into \( \mathbb{R}^2 \) given by
   \[
   f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2^2 y_1 - 4y_2 + 3,
   \]
   \[
   f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3.
   \]
   If \( \vec{a} = (0, 1) \) and \( \vec{b} = (3, 2, 7) \) then find the values of \( f(\vec{a}, \vec{b}) \).
8. Put \( f(0,0) = 0 \) and
   \[
   f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0)
   \]
   Show that \( D_{12} f \) and \( D_{21} f \) exist at every point of \( \mathbb{R}^2 \), but that \( D_{12} f \neq D_{21} f \) at \( (0, 0) \).

PART – B (4 x 12 = 48 Marks)
(Essay Answer Type)

9. a) i) Prove that the interval \((a, \infty)\) is measurable.
   ii) Let \( \{E_n\} \) be an infinite decreasing sequence of measurable sets, that is, a sequence with \( E_{n+1} \subset E_n \) for each \( n \). Let \( m(E_1) \) be finite. Then prove that
   \[
   m \left( \bigcap_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} m(E_n).
   \]
   OR

   b) i) Let \( f \) and \( g \) be two measurable real-valued functions defined on the same domain. Then prove that \( f + g \) is also measurable.
   ii) State and prove Littlewood's third principle.
10 a) i) Let \( f \) be a bounded function defined on \([a, b]\). If \( f \) is Riemann integrable on \([a, b]\) then prove that it is measurable and

\[
\mathcal{R} \int_a^b f(x) \, dx = \int_a^b f(x) \, dx.
\]

ii) Define integral of a nonnegative function. Prove that if \( f \) and \( g \) are nonnegative measurable functions then

\[
\int f + g = \int f + \int g
\]

OR

b) i) Let \( f \) be a nonnegative function which is integrable over a set \( E \). The prove that given \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for every set \( A \subset E \) with \( m(A) < \delta \) we have

\[
\int f < \varepsilon
\]

ii) Let \( \{f_n\} \) be a sequence of measurable functions that converges in measure to \( f \). Then prove that there is a subsequence \( \{f_{n_k}\} \) that converges to \( f \) almost everywhere.

11 a) Let \( f \) be an increasing real-valued function on the interval \([a, b]\). Then prove that \( f \) is differentiable almost everywhere. Also prove that the derivative \( f' \) is measurable and

\[
\int_a^b f'(x) \, dx \leq f(b) - f(a).
\]

OR

b) Prove that \( L^p \) spaces are complete.

12 a) State and prove the rank theorem.

OR

b) i) Suppose \( f \) is defined in an open set \( E \subset \mathbb{R}^2 \), and \( D_1 f \) and \( D_2 f \) exist at every point of \( E \). Suppose \( Q \subset E \) is a closed rectangle with sides parallel to the coordinate axes, having \((a, b)\) and \((a+h, b+k)\) as opposite vertices \(h \neq 0, k \neq 0\). Put

\[
\Delta(f, Q) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b).
\]

Then prove that there is a point \((x, y)\) in the interior of \( Q \) such that

\[
\Delta(f, Q) = hk(D_2 f)(x, y).
\]

ii) Prove that \( D_{21} f = D_{12} f \) if \( f \in C^2(E) \).

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