PART - A (8 x 4 = 32 Marks)
(Short Answer Type)

1. If \( f_1 \) and \( f_2 \) are linearly independent functions on an interval \( I \) then prove that \( f_1 + f_2 \) and \( f_1 - f_2 \) are also linearly independent on \( I \).

2. If \( \phi_1, \phi_2, ..., \phi_n \) are \( n \) linearly independent solutions and \( \phi \) is any solution of the \( n \)th order equation \( L(x(t)) - x^{(n)}(t) + b_1(t)x^{(n-1)}(t) + ... + b_n(t)x(t) = 0, t \in I \), then prove that there exists \( n \) constants \( c_1, c_2, ..., c_n \) such that \( \phi = c_1\phi_1 + ... + c_n\phi_n, t \in I \).

3. Write general \( n \)th order equation \( x^{(n)} = g(t, x, x', ..., x^{(n-1)}), t \in I \), into a system of first order \( n \) equations.

4. If \( \phi \) is a fundamental matrix for the system \( x' = A(t)x \) and if \( C \) is a constant non-singular matrix then prove that \( \phi C \) is also a fundamental matrix for above system.

5. Show that the function \( f(t, x) = x^2 + \cos^2 t \) satisfies Lipschitz condition in the region \( R : \{(t, x) : 0 \leq t \leq a, |x| \leq b\} \).

6. Define contraction mapping and also prove that every contraction mapping has unique fixed point.

7. Define equicontinuous and uniformly bounded of a family of function \( \{f_n(t)\} \) defined on \( I \) and also state Ascoli's Lemma.

8. If \( v, w \in C^1([t_0, t_0 + h], R) \) are lower and upper solutions of IVP \( x' = f(t, x), x(t_0) = x_0 \) and if \( f \) satisfies the inequality \( f(t, x) - f(t, y) \leq L(x-y) \) for \( x \geq y \) then prove that \( v(t_0) \leq w(t_0) \) implies \( v(t) \leq w(t) \) on \( I = [t_0, t_0 + h] \).

PART - B (4 x 12 = 48 Marks)
(Essay Answer Type)

9. a) State and prove Abel's formula for \( n \)th order linear homogeneous differential equation \( L(x)(t) = x^{(n)}(t) + b_1(t)x^{(n-1)}(t) + ... + b_n(t)x(t) = 0, t \in I \), where \( b_1, b_2, ..., b_n \) are continuous functions defined on an interval \( I \).

b) Solve \( x'' - 7x' = (3 - 36t)e^{4t} \) using the method of undetermined coefficients.
10 a) If $A(t)$ is an $n \times n$ matrix continuous on $I$ and if matrix $\Phi$ satisfies $X' = A(t)X$, $t \in I$ then prove that $\det \Phi$ satisfies the first order equation $(\det \Phi)' = (t \times A)(\det \Phi)$.

OR

b) Determine the fundamental matrix for the system $x' = Ax$, where

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

11 a) State and prove Picard's theorem for existence and uniqueness of solution of IVP $x' = f(t, x), x(t_0) = x_0$.

OR

b) Prove that IVP $x' = f(t, x), x(t_0) = x_0$ has unique solution on $[t_0, t_0 + h]$ using contraction principle if $f(t, x)$ is continuous on the strip $t_0 \leq t \leq t_0 + h$, $|x| < \infty$ and $f$ satisfies Lipschitz condition with Lipschitz constant $K > 0$.

12 a) If $f$ in IVP $x' = f(t, x), x(t_0) = x_0$, is non-increasing in $x$, then prove that the iterates $V_n(t)$ given by $V_{n+1} = f(t, V_n), V_{n+1}(t_0) = x_0$, and the unique solution $x(t)$ of above IVP satisfy the inequality $V_0(t) \leq V_1(t) \leq \ldots \leq x(t) \leq \ldots \leq V_1(t) \leq V_0(t)$, $t \in I$ provided $V_2(t) \geq V_0(t)$ and also the sequence $\{V_{2n}(t)\}, \{V_{2n+1}(t)\}$ converge uniformly to $\rho(t)$, $r(t)$ and $\rho(t) \leq x(t) \leq r(t)$, $t \in I$.

OR

b) Prove that IVP $x' = f(t, x), x(t_0) = x_0$ has unique solution on the strip $S = \{(t, x) ; t_0 \leq t \leq t_0 + h, |x| < \infty\}$ if $f$ is continuous and bounded on $S$ using Ascoli's lemma.