FACULTY OF SCIENCE
M. Sc. II – Semester Examination, May / June 2017
Subject: Maths / Applied Maths
Paper – I: Advanced Algebra

Time: 3 Hours  Max. Marks: 80

Note: Answer all questions from Part–A and Part–B. Each question carries 4 marks in Part–A and 12 marks in Part – B.

PART – A (8 x 4 = 32 Marks)
(Short Answer Type)

1. State and prove a Eisenstein criteria.
2. Show that $x^2 - 2$ is irreducible over $\mathbb{Q}$.
3. If $f(x)$ is an irreducible polynomial over $F$ then prove that $f(x)$ has a multiple root if and only if $f'(x) = 0$.
4. Let $p$ be a prime. Then prove that $f(x) = x^p - 1 \in \mathbb{Q}(x)$ has splitting field $\mathbb{Q}(\alpha)$ where $\alpha \neq 1$ and $\alpha^p = 1$.
5. If $E$ is a finite separable extension of a field $F$ and $H \leq G \left( \frac{E}{F} \right)$ then prove that $G \left( \frac{E}{H} \right) = H$ and $[E: E_H] = \left| G \left( \frac{E}{H} \right) \right|$.
6. Show that the Galois group of $x^4 - 1 + x^5$ is the Klein’s four group.
7. Show that $\Phi_4(x)$ and $x^4 - 1$ have the same Galois group.
8. If a > 0 is constructible then prove that $\sqrt{a}$ is constructible.

PART – B (4 x 12 = 48 Marks)
(Essay Answer Type)

9. (a) Let $E$ be an extension field of $F$ and $u \in E$ be algebraic over $F$. If $p(x) \in F(x)$ is a polynomial of least degree such that $q(u) = 0$ then prove that
   (i) $p(x)$ is irreducible over $F$.
   (ii) $p(x) \in F(x)$ is such that $u(p) = 0$ then $p(x) / q(x)$.
   (iii) There is exactly one monic polynomial $p(x) \in F(x)$ of least degree such that $p(u) = 0$.

   OR

   (b) Let $E$ be an algebraic extension of a field $F$ and $\sigma : F \rightarrow L$ be an embedding of $F$ into an algebraically closed field $L$. Then prove that $\sigma$ can be extended to an embedding $\eta : E \rightarrow L$.

10. (a) (i) Prove that the prime field of a field $F$ is either isomorphic to $\mathbb{Q}$ or to $\mathbb{F}_p$ where $p$ is a prime.
    (ii) If the multiplicative group $F^*$ of non zero element of a field $F$ is cyclic then prove that $F$ is finite.

    OR
(b) If $E$ is a finite extension of a field $F$ then prove that the following are equivalent.
   (i) $E = F(\alpha)$ for some $\alpha \in E$
   (ii) There are only a finite number of intermediate fields between $F$ and $E$.

11 (a) State and prove fundamental theorem of Galois theory.
   OR
   (b) State and prove fundamental theorem of algebra.

12 (a) Let $E$ be a finite extension of $F$ and $G \left( \frac{E}{F} \right)$ be a cyclic group of order $n$ generated by $\sigma$. If $w \in E$ is such that $w \sigma(w) \sigma^2(w) \cdots \sigma^{n-1}(w) = 1$ then prove that there exists $\alpha \in E^*$ such that $w = \sigma(\alpha)\sigma^{-1}$.
   OR
   (b) Prove that $f(x) \in F(x)$ is solvable by radicals over $F$ if and only if its splitting field $E$ over $F$ has solvable Galois group $G \left( \frac{E}{F} \right)$. 