FACULTY OF SCIENCE
M.Sc. IV-Semester Examination, May / June 2017
Subject: Mathematics
Paper - IV (a)
Banach Algebra

Time: 3 hours
Max. Marks: 80

Note: Answer all questions from Part-A and Part-B. Each question carries 4 marks in Part-A and 12 marks in Part-B.

PART - A (8 x 4 = 32 Marks)
(Short Answer Type)

1. In a normed algebra A, prove that (i) the multiplication \( x, y \rightarrow xy \) is jointly continuous in its factors and (ii) the product of two Cauchy sequences is Cauchy.

2. Let A be a Banach algebra with unity element. If \( z \in A \) and \( \|z\| < 1 \), then prove that \( 1 - z \) is invertible.

3. Let A be a Gelfand algebra. If M is a maximal ideal of A, then prove that M is closed.

4. Prove that every TDZ is singular.

5. If A is a C*-algebra with unity and if \( x \in A \) is self-adjoint, then prove that \( \sigma_A(x) \subset \mathbb{R} \).

6. In a C*-algebra with unity, if \( a \) is an element such that \(-a^* a \geq 0\), then prove that \( a = 0 \).

7. If A is a C*-algebra with unity and if \( a \in A \) is normal, then prove that \( \sigma(f(a)) = f(\sigma(a)) \) for all \( f \in \mathbb{C} \).

8. Let \( \|f\| \leq 1 \) and \( f \) is any complex polynomial, then prove that \( \|f(T)\| \leq \|f\| \cdot \|T\| \), where \( \Delta_1 = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \).

PART - B (4 x 12 = 48 Marks)
(Essay Answer Type)

9. a) Let A be a Banach algebra with unity. Let U be the set of all invertible elements of A. The prove the following:
   i) The mapping \( x \rightarrow x^{-1} \) (\( x \in U \)) is bicontinuous.
   ii) If \( h \) is a nonzero complex variable, then the limit \( \lim_{h \to 0} \frac{(x + h)^{-1} - x^{-1}}{h} \) exists and is equal to \( -x^2 \).

OR

b) i) Let A be a Banach algebra with unity element 1. Prove that for each \( x \in A \), \( p(x) \) is a proper subset of \( C \).
   ii) State and prove Gelfand-Mazur theorem.
10 a) State and prove Gelfand representation theorem.
    OR
    b) If $E$ is a Banach space and $T \in \mathcal{L}(E)$, then prove that the following conditions on $T$ are equivalent:
       i) $T$ is surjective;  ii) $T^*$ is bounded below.

11 a) Prove that a C*-algebra without unit may be embedded in a C*-algebra with unity.
    OR
    b) If $A$ is a C*-algebra with unity and if $f$ is a state on $A$, then prove that there exist a unital $*$-representation $\phi : A \rightarrow \mathcal{L}(H)$ and a vector $u \in H$ such that $f(a) = \langle \phi(a)u, u \rangle$ for all $a \in A$.

12 a) State and prove Gelfand-Naimark representation theorem.
    OR
    b) If $||T|| \leq 1$ and $f \in C(t; \Delta)$, then prove that there exists a sequence of complex polynomials $f_n$ such that $||f(T) - f_n(T)|| \rightarrow 0$. 