FACULTY OF SCIENCE  
M.Sc. II – Semester Examination, May / June 2018  

Subject: Mathematics  


Time : 3 Hours  
Max. Marks: 80

Note : Answer all questions from Part–A and Part–B. Each question carries 4 marks in Part–A and 12 marks in Part – B.

PART – A (8 x 4 = 32 Marks)  
(Short Answer Type)

1. Prove that $x^4$ and $x/x^3$ are linearly independent functions on $[0.1, 1]$ but they are linearly dependent on $[-0.1, 0]$ and $[0, 1]$.  

2. Prove that there are three linearly independent solution of the third order equation $x''' + b_1(t)x'' + b_2(t)x + b_3(t)x = 0$, $t \in I$ on an interval $I$.  

3. Let $h = \min \{ \frac{a_i}{b_i} \}$ then the successive approximations given by  
$$x_n(t) = x_0 + \int_{t_{n-1}}^{t_n} f(s, x_{n-1}(s)) ds, \quad n = 1, 2, 3, \ldots \quad \text{are valid on} \quad I = [t - t_0] \leq h. \quad \text{Further}$$  
$$|x(t) - x_0| = L |t - t_0| \leq h, \quad j = 1, 2, \ldots, \quad t \in I.$$

4. Prove the error $x(t) - x_n(t)$ satisfies the estimate $|x(t) - x_n(t)| \leq \frac{L(Kh)^{n+1}}{K(n+1)} e^{kh}$.  

5. Suppose that $f(t, x)$ is non-increasing in $x$ then prove that there exist lower and upper solutions $\nu(t)$, $\omega(t)$ such that $\nu(t) \leq \omega(t)$ on $I = [t_0, t_0 + h]$  


7. The equation $L_2(y) = a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ is self-adjoint if and only if $a_0(x) = a_1(x)$.  

8. If $y_1(x) = x$ is one solution of  
$$x^2y'' - xy' + y = 0, \quad x > 0 \quad \text{then find second solution } y_2(x).$$

PART – B (4 x 12 = 48 Marks)  
(Essay Answer Type)

9. (a) Let $b_1, b_2, \ldots, b_n$ be real or complex valued functions defined and continuous on an interval $I$ and $\phi_1, \phi_2, \ldots, \phi_n$ are $n$ solutions of the equation  
$L(x)(t) = x^{(n)}(t) + b_1x^{(n-1)}(t) + \ldots + b_n(t)x(t) = 0$ existing on $I$ then show that they are linearly independent on $I$ if and only if $L(t) \neq 0$ for every $t \in I$.  

(b) Solve $x'' - 4x' = te^{4t}$ by the method of undetermined coefficients.

10. (a) Let $x(t) = x(t, t_0, x_0)$ and $x^*(t) = x(t, t_0^*, x_0^*)$ be solutions of the IVPs $x' = f(t, x)$, $x(t_0) = x_0$ and $x^* = f(t, x)$, $x(t_0) = x_0^*$ respectively on an interval $a \leq t \leq b$. Let $f \in \text{Lip}(D, K)$ be bounded by $L$ in $D$ then show that for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $|x(t) - x^*(t)| < \epsilon$, $a \leq t \leq b$  
where $|t_0 - t_0^*| < \delta$ and $|x_0 - x_0^*| < \delta$.  

OR
(b) Prove that the IVP \( x' = f(t, x), \ x(t_0) = x_0 \) has a unique solution defined on 
\[ t_0 \leq t \leq t_0 + h, \quad h > 0 \] 
if the function \( f(t, x) \) is continuous in the strip \( t_0 \leq t \leq t_0 + h, \quad |x| < \infty \) and satisfies the Lipschitz condition.

\[ |f(t, x_1) - f(t, x_2)| \leq K |x_1 - x_2|, \quad K > 0 \] 
\( K \) being Lipschitz constant.

11 (a) Let \( V, W \in C([t_0, t_0 + h], R) \) be lower and upper solutions of \( x' = f(t, x), \ x(t_0) = x_0 \) respectively. Suppose that for \( x \geq y \), \( f \) satisfies the equality \( f(t, x) - f(t, y) \leq L (x - y) \). Where \( L \) is a positive constant then prove that \( v(t_0) \leq w(t_0) \) implies that \( v(t) \leq w(t) \), \( t \in [t_0, t_0 + h] \).

(b) Let \( f, v \in C(R^+, R^+), \ w \in C(R^+, R^+) \) and \( C > 0 \) satisfy

\[ f(t) \leq C + \int_{t_0}^{t} [v(s) f(s) + w(s, f(s))] ds, \quad t \geq t_0 \]

suppose, further that \( w(t, z \exp \int_{t_0}^{t} v(s) ds) \leq \lambda(t) g(z) \exp(\int_{t_0}^{t} v(s) ds) \) where \( \lambda, g \in C([0, \infty), (0, \infty)] \) and \( g(u) \) is non decreasing in \( u \) then prove that \( f(t) \leq G^{-1}[G(C) + \int_{t_0}^{t} \lambda(s) ds] \exp(\int_{t_0}^{t} v(s) ds), \quad t_0 \leq t \leq T \). Where \( G(u) - G(u_0) = \int_{u_0}^{u} \frac{ds}{g(s)} \), \( G^{-1}(u) \) is the inverse function of \( G(u) \) and \( T = \sup \{ t \geq t_0 / G(C) + \int_{t_0}^{t} v(s) ds \in \text{dom} G^{-1} \} \).

12 (a) State and prove Abel's formula.

(b) State and prove Bocher-Osgood theorem.