FACULTY OF SCIENCE
M.Sc. III – Semester Examination, January 2018
Subject: Mathematics
Paper – II: Functional Analysis

Time: 3 Hours
Max. Marks: 80

Note: Answer all questions from Part-A and Part-B. Each question carries 4 marks in Part-A and 12 marks in Part-B.

PART – A (8x4 = 32 Marks)
[Short Answer Type]

1. In a normed linear space prove that an open ball is a convex set.

2. Let $C[0,1]$ be the space of all real valued continuous functions defined on $[0,1]$ then prove that $C[0,1]$ is a normed linear space with the defined norm

$$||x|| = \max_{t \in [0,1]} |x(t)|, x \in C[0,1].$$


4. Prove that the space $\ell_p$ with $p \neq 2$ is not a Hilbert space.

5. If $\{A_n\}$ converge uniformly to $A$ then prove that $\{A_n\}$ is point wise convergent.

6. Let $A$ be an operator on a normed linear space $E_x$ then prove that

i) $||Ax|| \leq ||A|| \cdot ||x|| \quad \forall x \in E_x.$

ii) For every $\epsilon > 0$, there is an element $x_\epsilon$ such that $||Ax_\epsilon|| > (||A|| - \epsilon) \cdot ||x_\epsilon||.$

7. Define

i) Graph of an operator

ii) Closed linear operator

iii) Projection operator

8. Define self adjoint operator. Also prove that if $A, B$ are self adjoint operators then $A + B$ is also self adjoint.

PART – B (4x12 = 48 Marks)
[Essay Answer Type]

9. a) Prove that a subspace $X$ of a Banach space $E$ is complete if and only if the set $X$ is closed in $E.$

b) Let $Y$ and $Z$ be subspaces of a normed linear space $X$ and suppose that $Y$ is closed and is a proper subset $Z.$ Then for every real number $\theta \in (0,1)$ there is a $z \in Z$ such that $||z|| = 1, ||z - y|| \geq \theta \quad \forall y \in Y.$
10 a) Let $H$ be a Hilbert space and $L$ is closed, convex set in $H$ and $x \in H - L$ then there is a unique $y_0 \in L$ such that $||x - y_0|| = \inf_{y \in L} \{||x - y||\}$.

OR

b) Prove that a subspace $L$ of a Hilbert space $H$ is closed if and only if $L = L^\perp$.

11 a) State and approve generalized Hahn-Banach theorem.

b) State and prove uniform boundedness principle.

12 a) State and prove open mapping theorem.

b) Prove that every self adjoint operator $p$ satisfying $p^2 - p$ is an orthogonal projection on some subspace $L$ of the Hilbert space $H$. 