FACULTY OF SCIENCE
M.Sc. IV-Semester Examination, May / June 2018
Subject : MATHEMATICS
Paper - I
Advanced Complex Analysis

Time : 3 hours
Max. Marks : 80

Note: Answer all questions from Part-A and Part-B. Each question carries 4 marks in Part-A and 12 marks in Part-B.

PART - A (8 x 4 = 32 Marks)

1. If $z_1, z_2, \ldots, z_n$ are the zeros of $f$ inside the disc $D_R$ then prove that
   \[ \frac{R}{n(0)} \frac{dr}{r} = \sum_{k=1}^{N} \log \frac{R}{|z_k|} \]

2. Find the growth order of $\sin \pi z$.

3. Prove that the Gamma function extends to an analytic function in the half plane $\text{Re}(s) > 0$.

4. For $n \in \mathbb{N}$, prove that $\text{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}$.

5. Prove that $(\zeta(s))^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$.

6. If $\text{Re}(s) > 1$, prove that $\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m}$, where $p$ is prime, $m \in \mathbb{N}$.

7. For $M \in \text{SL}_2(\mathbb{R})$, prove that $f_M$ maps $H$ onto itself where $H$ is the upper half plane.

8. Prove that $\psi_\alpha(z) = \frac{\alpha - z}{1 - \alpha z}$, where $\alpha \in \mathbb{C}$, $|\alpha| < 1$ is an automorphism of the unit disc $D$.

PART - B (4 x 12 = 48 Marks)

9. a) Find the Hadamards products for
   i) $e^z - 1$  \hspace{2cm} ii) $\sin \pi z$
   OR
   b) State and prove Jensen's formula.

10. a) Prove that $\lim_{n \to \infty} \frac{n^s n!}{s(s+1) \ldots (s+n)} = \Gamma(s)$ for $s \neq 0, -1, -2, \ldots$.
    OR
    b) Prove that $\Gamma(s) \Gamma\left(s + \frac{1}{2}\right) = \sqrt{\pi} \ 2^{1-2s} \Gamma(2s)$.
11 a) Prove that, if \( \psi_1 \sim \frac{x^2}{x} \) as \( x \to \infty \), then prove that \( \psi(x) \sim x \) as \( x \to \infty \).

OR

b) Show that the function \( \zeta(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \) is real when \( s \) is real or when \( \Re(s) = \frac{1}{2} \).

12 a) State and prove Schwarz's lemma.

OR

b) Prove that every automorphism of upper half plane \( \mathbb{H} \) takes the form \( f_M \) for some \( M \in \text{SL}_2(\mathbb{R}) \).