FACULTY OF SCIENCE
M.Sc. IV – Semester Examination, May / June 2018
Subject: Mathematics
Paper – II
General Measure Theory

Time: 3 Hours
Max. Marks: 80

Note: Answer all questions from Part-A and Part-B.
Each question carries 4 marks in Part-A and 12 marks in Part-B.

PART – A (8x4 = 32 Marks)
[Short Answer Type]

1. Define a measure $\mu$ on a measurable space $(X, \mathcal{F})$. Prove that $\mu$ is countably sub additive.
2. State and prove Monotone convergence theorem.
3. Prove that countable union of positive sets is a positive set.
4. Suppose $(X, \mathcal{F}, \mu)$ is a measure space and $f$ is an integrable function on $X$ w.r.t. $\mu$.
   Prove that $\nu$ defined on $\mathcal{F}$ by $\nu(A) = \int_A f \, d\mu$ is a signed measure on $\mathcal{F}$.
5. Define a $\mu^*$-measurable set $E$. Suppose $\mu^*(E) = 0$, prove that $E$ is a $\mu^*$-measurable set.
6. Suppose $E \subset X$ and $x \in X$. Define $x -$ cross section of $E$ with usual notations. Prove that
   i) $\psi_{E_x}(y) = \psi_E(x, y) \quad \forall y \in y$
   ii) $E_x = \{E\}_x$
7. Suppose $\mu^*$ and $\mu$ are the outer and inner measures induced by a measure $\mu$ on an algebra $A$ of subsets of $X$.
   Prove that $\mu^*(E) \leq \mu^*(E) \quad \forall E \in P(x)$.
8. If $A \in A$ prove that
   $\mu(A) = \mu(A \cap E) + \mu^*(A \cap \bar{E})$.

PART – B (4x12 = 48 Marks)
[Essay Answer Type]

9 a) Suppose $(X, \mathcal{F}, \mu)$ is a measure space. Prove that it can be extended to a complete measure space $(X_1, \mathcal{F}_0, \mu_0)$ where $\beta \subset \beta_0$ and restriction $\mu_0$ to $\beta$ is $\mu$
   i.e. $\mu_0|\beta = \mu$.

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Suppose \((x, \beta)\) is a measurable space and \(E \in \beta\). Suppose \(f: E \rightarrow [-\infty, \infty]\) is a mapping. Prove that the following are equivalent:

i) \(\{x \in E: f(x) > \alpha\} \in \beta\) for all \(\alpha \in \mathbb{R}\)

ii) \(\{x \in E: f(x) \geq \alpha\} \in \beta\) for all \(\alpha \in \mathbb{R}\)

iii) \(\{x \in E: f(x) < \alpha\} \in \beta\) for all \(\alpha \in \mathbb{R}\)

iv) \(\{x \in E: f(x) \leq \alpha\} \in \beta\) for all \(\alpha \in \mathbb{R}\)

10 a) Suppose \(E\) is a measurable set such that \(0 < \nu(E) < \infty\). Prove that \(E\) has a positive set \(A\) such that \(\nu(A) > 0\).

b) State and prove Jordan – decomposition theorem.

11 a) Prove that the class \(\beta\) of all \(\mu^*\)-measurable sets is a \(\sigma\)-algebra of sets.

b) Suppose \((x, A, \mu)\) and \((y, \beta, \nu)\) are complete measure spaces and \(R\) is the class of all measurable rectangles in \(X \times Y\). Suppose \(E \in R_{\sigma\delta}\) with \((\mu \times \nu)(E) < \infty\). Prove that

i) The function \(g: x \rightarrow [0, \infty]\) defined by \(g(x) = \nu(E_x)\) \(\forall x \in X\) is a measurable function on \(x\). Also

ii) \(\int_X g \, d\mu = \int_X \nu(E_x) \, d\mu = (\mu \times \nu)(E)\).

12 a) Suppose \(E \subset X\) with \(\mu^*(E) < \infty\). Prove that \(E\) is \(\mu^*\)-measurable if and only if \(\mu^*(E) = \mu_*(E)\).

b) State and prove Coratheodory outer measure theorem.

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